ISRAEL JOURNAL OF MATHEMATICS 97 (1997), 203–219

THE KADEC–KLEE PROPERTY IN SYMMETRIC SPACES OF MEASURABLE OPERATORS

BY

V. I. CHILIN

Department of Mathematics, Tashkent State University
Tashkent 700095, Uzbekistan

AND

P. G. DODDS AND F. A. SUKOCHEV*

Department of Mathematics and Statistics, The Flinders University of South Australia
GPO Box 2100, Adelaide, SA 5001, Australia
e-mail: peter@ist.flinders.edu.au and sukochev@ist.flinders.edu.au

ABSTRACT

We show that if E is a separable symmetric Banach function space on the positive half-line then E has the Kadec–Klee property if and only if, for every semifinite von Neumann algebra $(\mathcal{M}, \tau)$, the associated space $E(\mathcal{M}, \tau)$ of τ-measurable operators has the Kadec–Klee property.

0. Introduction

If E is a normed linear space, then E is said to have the Kadec–Klee property if and only if sequential weak convergence on the unit sphere coincides with norm convergence. If E is a separable symmetric sequence space, then it was shown in [Ar] (see also [Si]) that E has the Kadec–Klee property if and only if the associated unitary matrix space C_E has the Kadec–Klee property. Here C_E is the space of all compact operators on l^2 for which $s(x) \in E$ with norm given by $\|x\|_{C_E} = \|s(x)\|_E$, where $s(x) = \{s_n(x)\}_1^\infty$ is the sequence of s-numbers of x. The purpose of this paper is to show that this result of Arazy continues to hold in the more general setting of symmetric spaces $E(\mathcal{M}, \tau)$ of measurable

* Research supported by the Australian Research Council.
Received January 2, 1994 and in revised form March 15, 1995

operators associated with a symmetric Banach function space on the positive half-line $\mathbb{R}^+$. Indeed, the principal result of the paper (Theorem 2.7) shows that the study of the Kadec–Klee property in the non-commutative spaces $E(\mathcal{M},\tau)$ of τ-measurable operators affiliated with a given semifinite von Neumann algebra $(\mathcal{M},\tau)$ may be reduced to the study of the same property in the (commutative) space E. Our methods are completely different to those employed by [Ar] in the trace ideal setting, and may be of independent interest. In particular, our approach is based on a systematic study of Kadec–Klee type properties in (commutative) rearrangement-invariant Banach function spaces [CDSS], and the study of (non-commutative) symmetric operator spaces using methods of real analysis [FK], [CS 1,2], [DDP1,2,3]. One new technical ingredient (Theorem 2.5) shows that if E is separable, then sequences in $E(\mathcal{M},\tau)$ which converge to zero for the measure topology contain subsequences that are approximately disjointly supported. This is an extension of the non-commutative setting of a well-known result of Kadec and Pelczynski [KP]. Our methods yield further theorems of Kadec–Klee type for symmetric operator spaces in which the weak topology is replaced by other natural topologies (Theorems 2.8, 2.9). Finally, via a renorming theorem of A. A. Sedaev [Se1,2], we show that the non-commutative space $E(\mathcal{M},\tau)$ always admits an equivalent Kadec–Klee norm, provided E is separable.

1. Preliminaries

In this section we collect some of the basic facts and notation that will be used in this paper. We denote by $\mathcal{M}$ a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$, with a fixed faithful and normal semifinite trace τ. The identity in $\mathcal{M}$ is denoted by 1. A linear operator x: $\mathrm{dom}(x) \to \mathcal{H}$, with domain $\mathrm{dom}(x) \subseteq \mathcal{H}$, is called **affiliated with** $\mathcal{M}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. The closed and densely defined operator x, affiliated with $\mathcal{M}$, is called τ-**measurable** if for every $\epsilon > 0$ there exists an orthogonal projection $p \in \mathcal{M}$ such the $p(\mathcal{H}) \subseteq \mathrm{dom}(x)$ and $\tau(1-p) < \epsilon$. The collection of all τ-measurable operators is denoted by $\widetilde{\mathcal{M}}$. With the sum and product defined as the respective closures of the algebraic sum and product, $\widetilde{\mathcal{M}}$ is a *-algebra. For $\epsilon, \delta > 0$ we denote by $N(\epsilon,\delta)$ the set of all $x \in \widetilde{\mathcal{M}}$ for which there exists an orthogonal projection $p \in \mathcal{M}$ such that $p(\mathcal{H}) \subseteq \mathrm{dom}(x)$, $\|xp\|_\infty \leq \epsilon$ and $\tau(1-p) \leq \delta$, where $\|\cdot\|_\infty$ denotes the usual operator norm. The sets $\{N(\epsilon,\delta): \epsilon,\delta > 0\}$ form a base at 0 for a metrizable Hausdorff topology in $\widetilde{\mathcal{M}}$, which is called the **measure**

topology. Equipped with this measure topology, $\widetilde{\mathcal{M}}$ is a complete topological *-algebra. These facts and their proofs can be found in the papers [Ne] and [Te]. For standard facts concerning von Neumann algebras, we refer to [SZ], [Ta].

We recall the notion of generalized singular value function [FK]. Given a self-adjoint operator x in $\mathcal{H}$ we denote by $e^x(\cdot)$ the spectral measure of x. Now assume that $x \in \widetilde{\mathcal{M}}$. Then $e^{|x|}(B) \in \mathcal{M}$ for all Borel sets $B \subseteq \mathbb{R}$, and there exists $s > 0$ such that $\tau(e^{|x|}(s,\infty)) < \infty$. For $x \in \widetilde{\mathcal{M}}$ and $t \geq 0$ we define

$$\mu_t(x) = \inf\{s \geq 0 \colon \tau(e^{|x|}(s,\infty)) \leq t\}.$$

The function $\mu(x)\colon [0,\infty) \to [0,\infty]$ is called the **generalized singular value function** (or decreasing rearrangement) of x; note that $\mu_t(x) < \infty$ for all $t > 0$. For the basic properties of this singular value function we refer the reader to [FK]; some additional properties can be found in [DDP1], [DDP2]. We note that a sequence $\{x_n\} \subseteq \widetilde{\mathcal{M}}$ converges to 0 for the measure topology if and only if $\mu_t(x) \to 0$ for all $t > 0$.

If we consider $\mathcal{M} = L^\infty(\mathbb{R}^+, m)$, where m denotes Lebesgue measure on $\mathbb{R}^+$, as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^+, m)$, with the trace given by integration with respect to m, it is easy to see that $\widetilde{\mathcal{M}}$ consists of all measurable functions on $\mathbb{R}^+$ which are bounded except on a set of finite measure, and that for $f \in \widetilde{\mathcal{M}}$, the generalized singular value function $\mu(f)$ is precisely the decreasing rearrangement of the function $|f|$ (and in this setting, $\mu(f)$ is frequently denoted by f^*). If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and τ is the standard trace, then it is not difficult to see that $\widetilde{\mathcal{M}} = \mathcal{M}$ and that the measure topology coincides with the operator norm topology. If $x \in \mathcal{M}$, then x is compact if and only if $\lim_{t\to\infty} \mu_t(x) = 0$; in this case,

$$\mu_n(x) = \mu_t(x), \quad t \in [n, n+1), \quad n = 0, 1, 2, \ldots,$$

and the sequence $\{\mu_n(x)\}_{n=0}^\infty$ is just the sequence of eigenvalues of $|x|$ in non-increasing order and counted according to multiplicity.

By $L^0(\mathbb{R}^+, m)$ we denote the space of all $\mathbb{C}$-valued Lebesgue measurable functions on $\mathbb{R}^+$ (with identification m-a.e.). A Banach space $(E, \|\cdot\|_E)$, where $E \subseteq L^0(\mathbb{R}^+, m)$, is called a **rearrangement-invariant Banach function space** if it follows from $f \in E, g \in L^0(\mathbb{R}^+, m)$ and $\mu(g) \leq \mu(f)$ that $g \in E$ and $\|g\|_E \leq \|f\|_E$. Furthermore, $(E, \|\cdot\|_E)$ is called a **symmetric Banach function space** if it has the additional property, that $f, g \in E$ and $g \prec\!\prec f$ imply that $\|g\|_E \leq \|f\|_E$. Here $g \prec\prec f$ denotes submajorization in the sense of Hardy–Littlewood–Polya:

$$\int_0^t \mu_s(g)ds \leq \int_0^t \mu_s(f)ds, \quad \text{for all } t > 0.$$

For the general theory of rearrangement-invariant Banach function spaces, we refer the reader to [KPS], [BS], [LT], although in the latter two references the class of function spaces considered is more restrictive.

Given a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ and a symmetric Banach function space $(E, \|\cdot\|_E)$ on $(\mathbb{R}^+, m)$ we define the corresponding non-commutative space $E(\mathcal{M}, \tau)$ by setting

$$E(\mathcal{M}, \tau) = \{x \in \widetilde{\mathcal{M}} \colon \mu(x) \in E\}.$$

Equipped with the norm $\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E$, the space $(E(\mathcal{M}, \tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ is a Banach space and is called the (non-commutative) **symmetric operator space** associated with $(\mathcal{M}, \tau)$ corresponding to $(E, \|\cdot\|_E)$. An extensive discussion of the various properties of such spaces can be found in [DDP1,2,3]. We shall frequently use the following submajorisation inequality: if $x, y \in \widetilde{\mathcal{M}}$ then

$$|\mu(x) - \mu(y)| \prec\prec \mu(x - y)$$

which is proved in [DDP1]. It follows in particular that if E is a symmetric Banach function space on $\mathbb{R}^+$ and if $x, y \in E(\mathcal{M}, \tau)$, then

$$\|\mu(x) - \mu(y)\|_E \leq \|x - y\|_{E(\mathcal{M},\tau)}.$$

If E is a symmetric Banach function space on $\mathbb{R}^+$, then E is separable if and only if the norm on E is order continuous in the sense that $0 \leq f_\tau \downarrow_\tau 0$ in E implies that $\|f_\tau\|_E \downarrow_\tau 0$, and in this case, the norm is order continuous (in the obvious sense) on the non-commutative space $E(\mathcal{M}, \tau)$. The symmetric space E is separable if and only if the Banach dual space E^* coincides with the (first) associate space (or Köthe dual) of E. In this case, the dual space E^* is a symmetric Banach function space on $\mathbb{R}^+$ and it follows from [DDP3] Theorems 5.6, 5.11 that the Banach dual of the space $(E(\mathcal{M}, \tau), \|\cdot\|_{E(\mathcal{M},\tau)})$ is the space $(E^*(\mathcal{M}, \tau), \|\cdot\|_{(E^*(\mathcal{M},\tau)})$. If E is separable, then E is **fully symmetric** in the sense that $f \in E, g \in L^0(\mathbb{R}^+, m)$ and $g \prec\prec f$ implies that $g \in E$ and $\|g\|_E \leq \|f\|_E$.

The following criterion for convergence in symmetric operator spaces is proved in [CS1] and [CS2], Theorem 5.2.

THEOREM 1.1: *Let E be a separable symmetric Banach function space on $\mathbb{R}^+$. If $x, x_n \in E(\mathcal{M},\tau), n = 1,2,\ldots$, then the following statements are equivalent:*

(i) $\|x_n - x\|_{E(\mathcal{M},\tau)} \to 0$.

(ii) $\|\mu(x_n) - \mu(x)\|_{E(\mathcal{M},\tau)} \to 0$ *and* $x_n \to x$ *weakly.*

(iii) $\|\mu(x_n) - \mu(x)\|_{E(\mathcal{M},\tau)} \to 0$ *and* $x_n \to x$ *in the measure topology.*

(iv) $\|\mu(x_n) - \mu(x)\|_{E(\mathcal{M},\tau)} \to 0$ *and* $\tau(x_n e) \to \tau(xe)$ *for all projections* $e \in \mathcal{M}$ *with* $\tau(e) < \infty$.

If $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$, then $\mathcal{N}$ will be called **proper** if the restriction $\tau_{\mathcal{N}}$ of τ to $\mathcal{N}$ is again semi-finite. The proof of the following result, which is essentially due to Ovčinnikov [Ov1,2], may be found in [DDP2], Theorem 3.5.

PROPOSITION 1.2: *If $0 \le x \in \widetilde{\mathcal{M}}$, and if $\lim_{t\to\infty} \mu_t(x) = 0$, then there exists a proper von Neumann subalgebra $M_x \subseteq L^\infty(\mathbb{R}^+)$ with $\mu(x) \in \widetilde{M_x}$, a proper commutative subalgebra $\mathcal{M}_x \subseteq \mathcal{M}$ and a positive rearrangement-preserving algebra $*$-isomorphism J_x of $\widetilde{M_x}$ onto $\widetilde{\mathcal{M}_x}$ whose restriction to the projections of M_x is a Boolean algebra isomorphism onto the projections of $\mathcal{M}_x$ and for which*

$$J_x(\mu(x)) = x.$$

If $\mathcal{N} \subseteq \mathcal{M}$ is a proper von Neumann subalgebra, then the conditional expectation

$$\mathcal{E}_{\mathcal{N}}: L^1(\mathcal{M},\tau) + \mathcal{M} \to L^1(\mathcal{N},\tau_{\mathcal{N}}) + \mathcal{N}$$

is defined as in the commutative setting via the equality

$$\tau_{\mathcal{N}}(\mathcal{E}_{\mathcal{N}}(x)y) = \tau(xy), \quad x \in L^1(\mathcal{M},\tau) + \mathcal{M}, \quad y \in L^1(\mathcal{N},\tau_{\mathcal{N}}) \cap \mathcal{N},$$

and an appeal to the fact that the spaces $L^1(\mathcal{N},\tau_{\mathcal{N}}) + \mathcal{N}, L^1(\mathcal{N},\tau_{\mathcal{N}}) \cap \mathcal{N}$ are dual in the sense of Köthe. See, for example, [DDP3] Theorem 5.6.

LEMMA 1.3: *Let $\mathcal{N} \subseteq \mathcal{M}$ be a proper von Neumann subalgebra. If E is a separable symmetric space on $\mathbb{R}^+$, then $\mathcal{E}_{\mathcal{N}}(x) \in E(\mathcal{N},\tau_{\mathcal{N}})$ for all $x \in E(\mathcal{M},\tau)$ and*

$$\tau_{\mathcal{N}}(\mathcal{E}_{\mathcal{N}}(x)y) = \tau(xy), \quad x \in E(\mathcal{M},\tau), \quad y \in E(\mathcal{N},\tau_{\mathcal{N}})^* = E^*(\mathcal{N},\tau_{\mathcal{N}}).$$

Proof: We observe that for each $x \in E(\mathcal{M},\tau)$, the mapping $y \to \tau(xy), y \in E(\mathcal{N},\tau_{\mathcal{N}})^*$, is a normal linear mapping in the sense of [DDP3] Definition 5.8

Consequently, by [DDP3] Theorem 5.11, there exists a unique element $\mathcal{E}(x) \in E^{*\times}(\mathcal{N}, \tau_{\mathcal{N}})$ such that

$$\tau_{\mathcal{N}}(\mathcal{E}(x)y) = \tau(xy), \quad x \in E(\mathcal{M}, \tau), \quad y \in E(\mathcal{N}, \tau_{\mathcal{N}})^*.$$

Here, $F^\times$ denotes the Köthe dual of F. It follows immediately that $\mathcal{E}$ coincides with the restriction of $\mathcal{E}_{\mathcal{N}}$ to $E(\mathcal{M}, \tau)$. Since E is separable, and $\mathcal{E}_{\mathcal{N}}(x) \prec\prec \mu(x) \in E$ for all $x \in E(\mathcal{M}, \tau)$, it follows that $\mathcal{E}(x) = \mathcal{E}_{\mathcal{N}}(x) \in E(\mathcal{N}, \tau_{\mathcal{N}})$ and this suffices to conclude the proof of the lemma. ∎

2. Kadec–Klee properties in symmetric operator spaces

The following is an immediate consequence of [CDSS] Propositions 2.1, 2.4.

PROPOSITION 2.1: *Let E be a separable symmetric Banach function space on $\mathbb{R}^+$ with the Kadec-Klee property. If $x, x_n \in E$, if $x \prec\prec x_n, n = 1, 2, \ldots$, and if $\|x_n\|_E \to \|x\|_E$, then the sequence $\{\mu(x_n)\}$ converges to $\mu(x)$ in measure.*

LEMMA 2.2: *Let E be a separable symmetric Banach function space on $\mathbb{R}^+$ with the Kadec–Klee property and let $x \in E(\mathcal{M}, \tau), \{x_n\} \subseteq E(\mathcal{M}, \tau)$. If $x_n \to x$ weakly and if $\|x_n\|_{E(\mathcal{M},\tau)} \to \|x\|_{E(\mathcal{M},\tau)}$, then $x_n \to x$ for the measure topology.*

Proof: If $x = u|x|$ is the polar decomposition of x then it is clear that $u^*x_n \to |x|$ for the weak topology $\sigma(E(\mathcal{M}, \tau), E(\mathcal{M}, \tau)^*)$. Let $\mathcal{N} = \mathcal{N}_{|x|}, M_{|x|}, J_{|x|}$ be as in the statement of Proposition 1.2. If $\mathcal{E}_{\mathcal{N}}$is the conditional expectation from $L^1(\mathcal{M}, \tau) + \mathcal{M}$ onto $L^1(\mathcal{N}), \tau_{\mathcal{N}}) + \mathcal{N}$, then it follows from Lemma 1.3 that $\mathcal{E}_{\mathcal{N}}(u^*x_n) \to |x|$ for the weak topology $\sigma(E(\mathcal{N}, \tau_{\mathcal{N}}), E^*((\mathcal{N}, \tau_{\mathcal{N}}))$. Since

$$\|\mathcal{E}_{\mathcal{N}}(z)\|_{E(\mathcal{N},\tau_{\mathcal{N}})} \leq \|z\|_{E(\mathcal{M},\tau)}$$

for all $z \in E(\mathcal{M}, \tau)$, it follows from a routine argument that

$$\|\mathcal{E}_{\mathcal{N}}(u^*x_n)\|_{E(\mathcal{N},\tau_{\mathcal{N}})} \to \|x\|_{E(\mathcal{M},\tau)} = \||x|\|_{E(\mathcal{N},\tau_{\mathcal{N}})}.$$

We denote by $\mathcal{E}_{|x|}$ the conditional expectation of $(L^1 + L^\infty)(\mathbb{R}^+)$ onto $(L^1 + L^\infty)(M_{|x|})$.

Since $E(\mathcal{N}, \tau_{\mathcal{N}})$ is isometrically isomorphic to $J_{|x|} \circ \mathcal{E}_{|x|}(E)$, it follows that $E(\mathcal{N}, \tau_{\mathcal{N}})$ has the Kadec–Klee property and so

$$\|\mathcal{E}_{\mathcal{N}}(u^*x_n) - |x|\|_{E(\mathcal{N},\tau_{\mathcal{N}})} \to 0,$$

and consequently, via [DDP1],

$$\|\mu(\mathcal{E}_{\mathcal{N}}(u^*x_n)) - \mu(|x|)\|_E \to 0.$$

For $n = 1, 2, \ldots$, set $h_n = \mu(\mathcal{E}_{\mathcal{N}}(u^*x_n)) - \mu(|x|)$ and observe that

$$\mu(x) \prec\!\!\prec \mu(h_n) + \mu(x_n), \quad n \geq 1.$$

Since

$$\|\mu(x) + \mu(h_n)\|_E \to \|\mu(x)\|_E,$$

it follows from Proposition 2.1 preceding that $\mu(x_n) + \mu(h_n) \to \mu(x)$ in measure, and since $\mu(h_n) \to 0$ in measure, it follows also that $\mu(x_n) \to \mu(x)$ in measure. Now suppose that $g \in E^*$ is such that $\|g\|_{E^*} = 1$ and $\|\mu(x)\|_E = \int_{[0,\infty)} \mu_t(x)\mu_t(g)dt$. Set

$$\phi(t) = \int_{[0,t)} \mu_s(g)ds, \quad t \geq 0,$$

and let Λ_ϕ be the corresponding Lorentz space on $\mathbb{R}^+$ consisting of all $f \in L^1 \cap L^\infty$ for which

$$\|f\|_{\Lambda_\phi} := \int_{[0,\infty)} \mu_t(f)\phi'(t)dt = \int_{[0,\infty)} \mu_t(f)\mu_t(g)dt < \infty.$$

Since E embeds continuously into Λ_ϕ with norm at most one, it follows that

$$\limsup_{n\to\infty} \|\mu(x_n)\|_{\Lambda_\phi} \leq \lim_{n\to\infty} \|\mu(x_n)\|_E = \|\mu(x)\|_{\Lambda_\phi}.$$

On the other hand, lower-semicontinuity of the norm on Λ_ϕ for convergence on measure implies that

$$\|\mu(x)\|_{\Lambda_\phi} \leq \liminf_{n\to\infty} \|\mu(x_n)\|_{\Lambda_\phi}.$$

Consequently

$$\|\mu(x_n)\|_{\Lambda_\phi} \to \|\mu(x)\|_{\Lambda_\phi},$$

and so by [CDSS] Corollary 1.3, it follows that

$$\|\mu(x_n) - \mu(x)\|_{\Lambda_\phi} \to 0.$$

Since Λ_ϕ embeds continuously into $L^1 + L^\infty$, it follows also that

$$\|\mu(x_n) - \mu(x)\|_{L^1+L^\infty} \to 0.$$

If F is the closure of $L^1 \cap L^\infty$ in $L^1 + L^\infty$, then F is separable and $\mu(x_n)$, $\mu(x) \in F$ for all $n \geq 1$. Since $x_n \to x$ pointwise on $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ and since $\|\mu(x_n) - \mu(x)\|_F = \|\mu(x_n) - \mu(x)\|_{L^1+L^\infty} \to 0$, it follows from the Theorem 1.1 that

$$\|x_n - x\|_F = \|x_n - x\|_{L^1+L^\infty} \to 0$$

and the assertion of the Lemma now follows. ∎

LEMMA 2.3: *Let $(\mathcal{M}, \tau), (\mathcal{N}, \sigma)$ be semifinite von Neumann algebras and let $0 \leq x, y \in \widetilde{\mathcal{M}}, 0 \leq u, v \in \widetilde{\mathcal{N}}$. If $xy = 0, \mu(x) \prec\!\prec \mu(u)$ and $\mu(y) \prec\!\prec \mu(v)$, then $\mu(x+y) \prec\!\prec \mu(u+v)$.*

Proof: Without loss of generality, we may assume that $\mathcal{M}$ contains no minimal projections. Otherwise we embed $\mathcal{M}$ into $\mathcal{M} \otimes L^\infty([0,1], dm)$ and observe that

$$\mu(x) = \mu(x \otimes 1)$$

where the decreasing rearrangement on the right is calculated relative to the tensor product of τ and the natural trace induced on $L^\infty([0,1], dm)$ by integration. By [FK] Lemma 4.1,

$$(1) \qquad \int_0^t \mu_s(z)ds = \sup\{\tau(eze) : e \in \mathcal{M}^p, \tau(e) \leq t\}, \quad 0 \leq z \in \widetilde{\mathcal{M}}, \quad t \geq 0.$$

For $t, \epsilon > 0$, it follows from (1) and $xy = 0$ that there exist disjoint projections e_1, e_2 in $\mathcal{M}$ and a real number $0 \leq \alpha \leq t$ such that

$$\tau(e_1) \leq \alpha, \qquad \tau(e_2) \leq t - \alpha$$

and

$$\int_0^t \mu_s(x+y)ds \leq \tau(e_1 x e_1) + \tau(e_2 y e_2) + \epsilon;$$

thus

$$(2) \qquad \int_0^t \mu_s(x+y)ds \leq \int_0^\alpha \mu_s(u)ds + \int_0^{t-\alpha} \mu_s(v)ds + \epsilon.$$

Again from (1), there exist projections e_3, e_4 in $\mathcal{M}$ such that

$$\tau(e_3) \le \alpha, \qquad \tau(e_4) \le t - \alpha$$

and

$$\int_0^\alpha \mu_s(u)ds + \int_0^{t-\alpha} \mu_s(v)ds \le \tau(e_3ue_3) + \tau(e_4ve_4) + \epsilon. \tag{3}$$

Set $e = e_3 \vee e_4$, then $\tau(e) \le t$ and

$$\tau(e_3ue_3) + \tau(e_4ve_4) \le \tau(e(u+v)e) \le \int_0^t \mu_s(u+v)ds. \tag{4}$$

The lemma follows now from (2), (3) and (4). ∎

LEMMA 2.4: *If $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra, and if e, f are projections in $\mathcal{M}$ with $\tau(f) < \infty$, then*

$$\tau(f \wedge (1-e)) \ge \tau(f) - \tau(e).$$

The proof of the Lemma follows immediately from the fact that the projections $f - f \wedge (1-e)$, $f \vee (1-e) - (1-e)$ are equivalent, together with the equality

$$f \vee (1-e) - (1-e) = (f \vee (1-e)) \wedge e.$$

We show now that sequences converging to zero for the measure topology contain subsequences that are approximately both right and left disjointly supported. For the case that $\tau(1) < \infty$, the theorem which follows is proved in [Su], and in the case that $\mathcal{M}$ is commutative is due to Kadec and Pelczynski [KP]. See also [KSS].

THEOREM 2.5: *Let E be a separable symmetric space on $\mathbb{R}^+$. If the sequence $\{x_n\} \subseteq E(\mathcal{M}, \tau)$ converges to zero in the measure topology, then there exists a subsequence $\{y_n\} \subseteq \{x_n\}$ and sequences $\{p_n\}, \{q_n\}$ of mutually orthogonal projections in $\mathcal{M}$ such that*

$$\|y_n - q_n y_n p_n\|_{E(\mathcal{M},\tau)} \to 0.$$

Proof: By [Su], Proposition 2.2, we may assume that $\tau(1) = \infty$. We assume first that $x_n \ge 0, n \ge 1$. Setting $e_n = \chi_{[2^{-n},\infty)}(x_n)$, $n \ge 1$, and passing to a subsequence if necessary, we may assume that

$$\tau(e_n) < 2^{-n}, \quad x_n e_n = e_n x_n \ge 2^{-n} e_n, \quad x_n e_n^\perp = e_n^\perp x_n \le 2^{-n} e_n^\perp$$

for all $n \geq 1$ where $f^{\perp} = 1 - f$. We set $e = \bigvee_{n \geq 1} e_n$. Note that $\tau(e) \leq 1$. Passing to a further subsequence if necessary and relabelling, it follows from the proof of [Su], Proposition 2.2 applied to the finite von Neumann algebra $e\mathcal{M}e$, that there exists a sequence $\{p_n'\}$ of mutually orthogonal projections in $\mathcal{M}$ with $0 \leq p_n' \leq e$ for all $n \geq 1$, such that

$$\|x_n e_n - x_n e_n p_n'\|_{E(\mathcal{M},\tau)} \to 0.$$

We may assume that $\|x_n e_n^{\perp}\|_{E(\mathcal{M},\tau)} \not\to 0$. By separability of E, there exist projections $0 \leq f_n \leq e_n^{\perp}$ such that $f_n x_n = x_n f_n, \tau(f_n) < \infty$ and

$$\|x_n e_n^{\perp} - x_n f_n\|_{E(\mathcal{M},\tau)} < 2^{-n}, \quad n \geq 1.$$

Further, the estimate

$$\|f_n x_n\|_{E(\mathcal{M},\tau)} \leq 2^{-n} \|\chi_{[0,\tau(f_n))}\|_E$$

shows that $\tau(f_n) \to \infty$. We now define a subsequence $\{n(k)\}$ of natural numbers and sequences $\{p_k''\}_1^{\infty}, \{r_k\}_1^{\infty}$ of projections in $\mathcal{M}$ as follows. Choose $n(1)$ such that

$$\tau(f_{n(1)}) > \tau(e) \quad \text{and} \quad \|\mu(x_{n(1)} f_{n(1)}) \chi_{[0,\tau(e))}\|_E < 2^{-1}.$$

Set $r_1 = e$ and let $p_1'' = f_{n(1)} \wedge (1 - e)$. Observe that Lemma 2.4 implies that $p_1'' \neq 0$. If natural numbers $n(j)$ and projections p_j'', e_j have been defined, $1 \leq j < k$, set $r_k = e \vee p_{n(1)}'' \vee \cdots \vee p_{k-1}''$. Choose $n(k)$ such that

$$\tau(f_{n(k)}) > \tau(r_k) \quad \text{and} \quad \|\mu(x_{n(k)} f_{n(k)}) \chi_{[0,\tau(r_k))}\|_E < 2^{-k},$$

and set $p_k'' = f_{n(k)} \wedge (1 - r_k)$. This choice is possible because $0 \leq x_n f_n \leq 2^{-n} f_n,\ n \geq 1$. From Lemma 2.4, it follows that

$$\tau(f_{n(k)} - p_k'') = \tau(f_{n(k)} - f_{n(k)} \wedge (1 - r_k)) < \tau(r_k), \quad k \geq 1$$

and this implies as well that $p_k'' \neq 0$. It is clear that $p_n'' p_m'' = 0$ for all $n, m \geq 1, n \neq m$ and, since $p_n'' \leq 1 - e$, it follows also that $p_n'' p_m' = 0$, for all $n, m \geq 1, n \neq m$. We obtain that

$$\begin{aligned}
\|x_{n(k)} e_{n(k)}^{\perp} - x_{n(k)} e_{n(k)}^{\perp} p_k''\|_{E(\mathcal{M},\tau)} &\leq 2^{-k} + \|x_{n(k)} f_{n(k)} - x_{n(k)} p_k''\|_{E(\mathcal{M},\tau)} \\
&\leq 2^{-k} + \|\chi_{[0,\tau(f_{n(k)} - p_k''))} \mu(x_{n(k)} f_{n(k)})\|_E \\
&\leq 2^{-k} + \|\chi_{[0,\tau(r_k))} \mu(x_{n(k)} f_{n(k)})\|_E \\
&\leq 2^{-k+1}, \quad k \geq 1.
\end{aligned}$$

By suitable relabelling, we may assume that $\|x_n e_n^\perp - x_n e_n^\perp p_n''\|_{E(\mathcal{M},\tau)} \to 0$. If we set $p_n = p_n' + p_n''$ and observe that

$$x_n p_n = (x_n e_n + x_n e_n^\perp)(p_n' + p_n'') = x_n e_n p_n' + x_n e_n p_n'', \quad n \geq 1,$$

we obtain that $p_n p_m = 0$ for all $n, m \geq 1, n \neq m$ and

$$\|x_n - x_n p_n\|_{E(\mathcal{M},\tau)} \to 0.$$

The assertion of the theorem now follows by the same argument as in the finite trace setting. We include the details for the sake of completeness. Suppose that $\{x_n\} \subseteq E(\mathcal{M},\tau)$ converges to 0 for the measure topology. By the first part of the proof applied to the sequence $\{|x_n|\}$, and passing to a subsequence if necessary, there exists a sequence $\{p_n\}$ of mutually orthogonal projections such that $\|\,|x_n| - |x_n|p_n\|_{E(\mathcal{M},\tau)} \to 0$. From the polar decomposition, it follows that

$$\|x_n - x_n p_n\|_{E(\mathcal{M},\tau)} \leq \|\,|x_n| - |x_n|p_n\|_{E(\mathcal{M},\tau)} \to 0.$$

Again passing to a subsequence and relabelling, there exists a sequence $\{q_n\}$ of mutually orthogonal projections such that $\|p_n x_n^* - p_n x_n^* q_n\|_{E(\mathcal{M},\tau)} \to 0$, and the assertion of the theorem now follows. ∎

If $x \in \widetilde{\mathcal{M}}$, then the right and left support projections of x are denoted by $r(x), l(x)$ respectively. Recall that if $x = u|x|$ is the polar decomposition, then $u^*u = r(x)$ and $uu^* = l(x)$ (see, for example, [Ta]). If $\{x_n\} \in \widetilde{\mathcal{M}}$, then $\{x_n\}$ is said to be **right** (respectively, **left**) **disjointly supported** if and only if $r(x_n)r(x_m) = 0$ (respectively, $l(x_n)l(x_m) = 0$) for all $m \neq n$.

LEMMA 2.6: *Let E be a separable symmetric Banach function space on $\mathbb{R}^+$. Assume that $x \in E(\mathcal{M},\tau)$, that $\{y_n\} \subseteq E(\mathcal{M},\tau)$ is both right and left disjointly supported and that $y_n \to 0 \quad \sigma(E(\mathcal{M},\tau), E(\mathcal{M},\tau)^*)$. There exists $f \in E$ and a sequence $\{f_n\} \subseteq E$ such that*

(i) $f f_n = f_n f_m = 0, \quad n, m \geq 1, n \neq m$;
(ii) $\mu(f) = \mu(x), \quad \mu(f_n) = \mu(y_n), \quad n \geq 1$;
(iii) $f_n \to 0 \quad \sigma(E, E^*)$;
(iv) $\|f_n + f\|_E - \|x + y_n\|_{E(\mathcal{M},\tau)} \to 0$.

Proof: We let $\mathbb{R}^+ = \bigcup_{n=0}^\infty I_n$ where $I_n \cap I_j = \emptyset, n \neq j, m(I_n) = \infty, n, j \geq 0$ and each I_n is a countable disjoint union of semi-open intervals of the form $[a, b)$. Let $\phi_n \colon I_n \to \mathbb{R}^+$ be a measure preserving bijection for each $n \geq 0$. We set

$$f := \mu(x) \circ \phi_0, \quad f_n := \mu(y_n) \circ \phi_n, \quad n \geq 1.$$

It is clear that (i) and (ii) are satisfied. To show (iii), let $g \in E^*$ and set $g_n = g\chi_{I_n}, n \geq 1$. For each $n \geq 1$, we let $\mathcal{M}_n = \mathcal{M}_{|y_n|}, M_n = M_{|y_n|}, J_n = J_{|y_n|}$ be as given by Proposition 1.2, and let $\mathcal{E}_n$ be the conditional expectation of $(L^1 + L^\infty)(\mathbb{R}^+)$ onto $(L^1 + L^\infty)(M_n)$. We observe that

$$\begin{aligned}\int_{[0,\infty)} f_n g dm &= \int_{[0,\infty)} \mu(y_n) \circ \phi_n g_n dm \\ &= \int_{[0,\infty)} \mu(y_n)\mathcal{E}_n(g_n \circ \phi_n^{-1}) dm \\ &= \tau\left(J_n(\mu(y_n))J_n(\mathcal{E}_n(g_n \circ \phi_n^{-1}))\right) \\ &= \tau(|y_n|z_n),\end{aligned}$$

where $z_n = J_n(\mathcal{E}_n(g \circ \phi_n^{-1})), n \geq 1$. For $n \geq 1$, let $y_n = u_n|y_n|$ be the polar decomposition. Observe that

$$\mu(r(y_j)z_j u_j^*) \leq \mu(z_j) \prec\prec \mu(g_j), \quad j \geq 1.$$

Since

$$l(|r(y_j)z_j u_j^*|) = r(r(y_j)z_j u_j^*) \leq l(y_j), \quad j \geq 1,$$

it follows from Lemma 2.3 that

$$\sum_{j=1}^N |r(y_j)z_j u_j^*| \prec\prec \sum_{j=1}^N g_j \leq g, \quad N \geq 1,$$

and consequently

$$\|\sum_{j=1}^N |r(y_j)z_j u_j^*|\|_{E^*(\mathcal{M},\tau)} \leq \|g\|_{E^*(\mathcal{M},\tau)}, \quad N \geq 1.$$

It follows from [DDP3] Corollary 5.12 that

$$w = \sup_N \sum_{j=1}^N |r(y_j)z_j u_j^*|$$

exists in $E^*(\mathcal{M},\tau)$ and that

$$\tau(yw) = \sum_{j\geq 1} \tau(y|r(y_j)z_j u_j^*|), \quad y \in E(\mathcal{M},\tau).$$

We let

$$r(y_j)z_ju_j^* = v_j|r(y_j)z_ju_j^*|, \quad j \geq 1$$

be the polar decomposition, set $v = \sum_{j\geq 1} v_j$, with convergence for the strong operator topology, and let $z = vw \in E^*(\mathcal{M}, \tau)$. Since $v_jv_j^* \leq r(y_j)$, it follows that $y_nv = y_nv_n$, for all $n \geq 1$, and since $v_j^*v_j \leq l(y_j)$, $j \geq 1$, it follows that $v_n|r(y_j)z_ju_j^*| = 0, n \neq j$. Consequently,

$$\begin{aligned}\tau(y_nz) &= \tau(y_nv_nw)\\ &= \sum_{j=1}^{\infty} \tau(y_nv_n|r(y_j)z_ju_j^*|)\\ &= \tau(y_nr(y_n)z_nu_n^*)\\ &= \tau(|y_n|z_n) = \int_{[0,\infty)} f_ngdm,\end{aligned}$$

and the assertion of (iii) now follows.

To establish (iv), if

$$p_n = \sup_{i\geq n} l(y_i), \quad q_n = \sup_{i\geq n} r(y_i), \quad n \geq 1,$$

then $p_n^\perp \uparrow_n 1, q_n^\perp \uparrow_n 1$ and it follows from [CS1] that

$$\|x - p_n^\perp xq_n^\perp\|_{E(\mathcal{M},\tau)} \to 0.$$

Since

$$|p_n^\perp xq_n^\perp + y_n| = |p_n^\perp xq_n^\perp| + |y_n|$$

it follows easily that

$$\mu(p_n^\perp xq_n^\perp + y_n) = \mu(|p_n^\perp xq_n^\perp| + |y_n|) = \mu\left(\mu(p_n^\perp xq_n^\perp) \circ \phi_0 + f_n\right)$$

and so

$$\|p_n^\perp xq_n^\perp + y_n\|_{E(\mathcal{M},\tau)} = \|\mu(p_n^\perp xq_n^\perp) \circ \phi_0 + f_n\|_E.$$

It follows that

$$\begin{aligned}&|\|x + y_n\|_{E(\mathcal{M},\tau)} - \|f_n + f\|_E|\\ &\leq \|x - p_n^\perp xq_n^\perp\|_{E(\mathcal{M},\tau)} + \||\mu(p_n^\perp xq_n^\perp) - \mu(x)| \circ \phi_0\|_E\\ &\leq 2\|x - p_n^\perp xq_n^\perp\|_{E(\mathcal{M},\tau)}\end{aligned}$$

and the statement of (iv) now follows. ∎

We remark that if in Lemma 2.6 preceding, the assumption that $y_n \to 0$ weakly is replaced by the assumption that $y_n \to 0$ for the measure topology, then the conclusion of the Lemma continues to hold if assertion (iii) is replaced by the statement that $f_n \to 0$ in measure.

We may now state the principal result of this paper.

THEOREM 2.7: *If E is a separable symmetric Banach function space on $\mathbb{R}^+$, then E has the Kadec–Klee property if and only if $E(\mathcal{M},\tau)$ has the Kadec–Klee property for every semifinite von Neumann algebra $(\mathcal{M},\tau)$.*

Proof: Let $x \in E(\mathcal{M},\tau), \{x_n\} \subseteq E(\mathcal{M},\tau)$. Suppose that $x_n \to x$ weakly and $\|x_n\|_{E(\mathcal{M},\tau)} \to \|x\|_{E(\mathcal{M},\tau)}$. It follows from Lemma 2.2 that $x_n \to x$ for the measure topology and so passing to a subsequence and relabelling if necessary, and appealing to Theorem 2.5, we may assume that there exists a right and left disjointly supported sequence $y_n \subseteq E(\mathcal{M},\tau)$ such that $x_n = x + y_n, n \geq 1$. Let $f, \{f_n\}$ be as in the statement of Lemma 2.6 and note that $f_n + f \to f$ for the weak topology $\sigma(E, E^*)$. Since

$$\|x + y_n\|_{E(\mathcal{M},\tau)} = \|x_n\|_{E(\mathcal{M},\tau)} \to \|x\|_{E(\mathcal{M},\tau)},$$

it follows from Lemma 2.6(iv) that $\|f + f_n\|_E \to \|f\|_E$. Since E has the Kadec–Klee property, it follows that $\|f_n\|_E \to 0$. Consequently,

$$\|x_n - x\|_{E(\mathcal{M},\tau)} = \|y_n\|_{E(\mathcal{M},\tau)} = \|f_n\|_E \to 0,$$

and this completes the proof of the theorem. ∎

In the trace ideal setting, the preceding theorem is due to Arazy [Ar].

Let E be a separable Banach function space on $\mathbb{R}^+$. The space $E(\mathcal{M},\tau)$ is said to have the Kadec–Klee property with respect to $L^1(\mathcal{M},\tau) \cap \mathcal{M}$ (respectively, the measure topology) if and only if $x, x_n \in E(\mathcal{M},\tau), n = 1,2,\ldots, x_n \to x$ for the weak topology $\sigma(E(\mathcal{M},\tau), L^1(\mathcal{M},\tau) \cap \mathcal{M})$ (respectively for the measure topology), $\|x_n\|_{E(\mathcal{M},\tau)} \to \|x\|_{E(\mathcal{M},\tau)}$ imply $\|x_n - x\|_{E(\mathcal{M},\tau)} \to 0$. We remark that if E has the Kadec–Klee property with respect to $L^1 \cap L^\infty$, then evidently E has the Kadec–Klee property. However, the converse is not valid [CDSS], Example 2.8 and Theorem 2.10. With the obvious notational changes, the proof of the preceding Theorem immediately yields the following result.

THEOREM 2.8: *If E is a separable symmetric Banach function space on $\mathbb{R}^+$, then E has the Kadec–Klee property with respect to $L^1 \cap L^\infty$ if and only if $E(\mathcal{M},\tau)$ has the Kadec–Klee property with respect to $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ for every semifinite von Neumann algebra $(\mathcal{M},\tau)$.*

In the trace ideal setting, the preceding theorem is noted in [Ar], with a more direct proof being given in Simon [Si].

THEOREM 2.9: *If E is a separable symmetric Banach function space on $\mathbb{R}^+$, then E has the Kadec–Klee property with respect to the measure topology if and only if $E(\mathcal{M},\tau)$ has the Kadec–Klee property with respect to the measure topology for every semifinite von Neumann algebra $(\mathcal{M},\tau)$.*

The preceding theorem follows easily from Theorem 2.5 and the remark following Lemma 2.6. We leave further details to the interested reader. The preceding Theorem 2.9 may also be found in [CS1], where it is proved by other methods. The present approach via Theorem 2.5 is not only somewhat simpler that of [CS1] but provides a unified approach to the wider class of Kadec–Klee properties considered here. Let us mention that it follows, in particular, from Theorem 2.9 that the spaces $L^p(\mathcal{M},\tau)$, $1\le p<\infty$ have the Kadec–Klee property for the measure topology. This result was first proved by Fack and Kosaki ([FK], Theorem 3.7).

If E is a separable symmetric Banach function space on $\mathbb{R}^+$, then it has been shown by A.A. Sedaev [Se1,2] that there exists an equivalent symmetric norm $\|\cdot\|_0$ on E such that $(E,\|\cdot\|_0)$ has the Kadec–Klee property with respect to $L^1\cap L^\infty$; in particular, $(E,\|\cdot\|_0)$ also has the Kadec–Klee property, and from [CDSS], Proposition 1.7, it follows further that $(E,\|\cdot\|_0)$ has the Kadec–Klee property for convergence in measure. For the case of the usual Kadec–Klee property in symmetric function spaces on the interval $[0,1]$, see also [DGL]. It follows from Theorems 2.7, 2.8, 2.9 that Sedaev's renorming theorem immediately carries over to the non-commutative setting.

COROLLARY 2.10: *If E is a separable symmetric Banach function space on $\mathbb{R}^+$, then $F(\mathcal{M},\tau)$ admits an equivalent norm $\|\cdot\|'$ such that $(E(\mathcal{M},\tau),\|\cdot\|')$ has the Kadec–Klee property with respect to $L_1(\mathcal{M},\tau)\cap\mathcal{M}$, and the Kadec–Klee property with respect to convergence in measure. In particular, $(E(\mathcal{M},\tau),\|\cdot\|')$ has the Kadec–Klee property.*

We mention one further consequence of Theorem 2.9. Let τ be the standard trace on $\mathcal{L}(\mathcal{H})$. If Λ_ϕ is a separable Lorentz space on $\mathbb{R}^+$, then Λ_ϕ has the Kadec–Klee property for convergence in measure, by [CDSS] Corollary 1.3. If we denote by $\mathcal{C}_\phi$ the trace ideal $\Lambda_\phi(\mathcal{L}(\mathcal{H}), \tau)$, then Theorem 2.9 implies that $\mathcal{C}_\phi$ has the Kadec–Klee property for convergence in measure. Since the measure topology coincides with the operator norm topology in this special case, we obtain the following characterisation of norm convergence in $\mathcal{C}_\phi$.

COROLLARY 2.11: *If Λ_ϕ is a separable Lorentz space on $\mathbb{R}^+$, and if $x, x_n \in \mathcal{C}_\phi, n \geq 1$, then the following statements are equivalent:*

(i) $\|x_n - x\|_{\mathcal{C}_\phi} \to 0$.

(ii) $\|x_n - x\|_\infty \to 0$ *and* $\|x_n\|_{\mathcal{C}_\phi} \to \|x\|_{\mathcal{C}_\phi}$.

References

[Ar] J. Arazy, *More on convergence in unitary matrix spaces*, Proceedings of the American Mathematical Society **83** (1981), 44–48.

[BS] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, New York, 1988.

[CDSS] V. I. Chilin, P. G. Dodds, A. A. Sedaev and F. A. Sukochev, *Characterization of Kadec–Klee properties in symmetric spaces of measurable functions*, preprint, 1994.

[CS1] V. I. Chilin and F. A. Sukochev, *Measure convergence in regular non-commutative symmetric spaces*, Izv. VUZov (Matematika) **9** (1990), 63–70 (Russian); English translation: Soviet Mathematics **34** (1990), 78–87.

[CS2] V. I. Chilin and F. A.Sukochev, *Weak convergence in non-commutative symmetric spaces*, Journal of Operator Theory **31** (1994), 35–65.

[DDP1] P. G. Dodds, T. K. Dodds and B. de Pagter, *Non-commutative Banach function spaces*, Mathematische Zeitschrift **201** (1989), 583–597.

[DDP2] P. G. Dodds, T. K. Dodds and B. de Pagter, *Fully symmetric operator spaces*, Integral Equations and Operator Theory **15** (1992), 942–972.

[DDP3] P. G. Dodds, T. K. Dodds and B. de Pagter, *Non-commutative Köthe duality*, Transactions of the American Mathematical Society **339** (1993), 717–750.

[DGL] W. J. Davis, N. Ghoussoub and J. Lindenstrauss, *A lattice renorming theorem and applications to vector-valued processes*, Transactions of the American Mathematical Society **263** (1981), 531–540.

[FK] T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pacific Journal of Mathematics **123** (1986), 269–300.

[KP] M. I. Kadec and A. Pelczynski, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Mathematica **21** (1962), 161–176.

[KPS] S. G. Krein, Ju. I. Petunin and E. M. Semenov, *Interpolation of linear operators,* Translations of Mathematical Monographs, Vol. 54, American Mathematical Society, 1982.

[KSS] A. V. Krygin, F. A. Sukochev and V. E. Sheremetjev, *Convergence in measure, weak convergence and structure of subspaces in symmetric spaces of measurable operators,* Dep. VINITI, N2487-B92, 1–34 (Russian).

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II,* Springer-Verlag, Berlin, 1979.

[Ne] E. Nelson, *Notes on non-commutative integration,* Journal of Functional Analysis **15** (1974), 103–116.

[Ov1] V. I. Ovčinnikov, *s-numbers of measurable operators,* Funktsional'nyi Analiz i Ego Prilozheniya **4** (1970), 78–85 (Russian).

[Ov2] V. I. Ovčinnikov, *Symmetric spaces of measurable operators,* Dokl. Nauk SSSR **191** (1970), 769–771 (Russian); English Translation: Soviet Math. Dokl. **11** (1970), 448–451.

[Se1] A. A. Sedaev, *On the (H)-property in symmetric spaces,* Teoriya funkcii, Func. Anal. i Prilozenia **11** (1970), 67–80 (Russian).

[Se2] A. A. Sedaev, *On weak and norm convergence in interpolation spaces,* Trudy 6 zimney shkoly po mat. programm. i smezn. voprosam. Moscow, 1975, pp. 245–267 (Russian).

[Si] B. Simon, *Convergence in trace ideals,* Proceedings of the American Mathematical Society **83** (1981), 39–43.

[Su] F. A. Sukochev, *Non-isomorphism of L_p-spaces associated with finite and infinite von Neumann algebras,* Proceedings of the American Mathematical Society (to appear).

[SZ] S. Stratila and L. Zsido, *Lectures on Von Neumann Algebras,* Editura and Abacus Press, 1979.

[Ta] M. Takesaki, *Theory of Operator Algebras I,* Springer-Verlag, New York–Heidelberg–Berlin, 1979.

[Te] M. Terp, *L^p-spaces associated with von Neumann algebras,* Notes, Copenhagen University, 1981.